Consider the non-homogeneous linear differential equation $2x^2y'' + 7xy' - 3y = \frac{A}{x}$.

SCORE: _____ / 11 PTS

[a] If $y = \frac{2}{x}$ is a particular solution of the equation, find the value of A.

$2x^2(4x^{-3}) + 7x(-2x^{-2}) - 3(2x^{-1}) = 8x^{-1} - 14x^{-1} - 6x^{-1}$

$= -12x^{-1}$ $\quad A = -12$

[b] **Using linearity**, find a particular solution of $2x^2y'' + 7xy' - 3y = \frac{4}{x}$.

$\frac{4}{x} = -\frac{1}{3}\left(\frac{-12}{x}\right)$

$y_p = -\frac{1}{3}\left(\frac{2}{x}\right) = -\frac{2}{3}x^{-1}$

1 POINT EACH UNLESS OTHERWISE NOTED

[c] **Using superposition**, find the general solution of $2x^2y'' + 7xy' - 3y = \frac{4}{x}$.

$2r^2 + 5r - 3 = 0$

$(2r-1)(r+3) = 0$

$r = \frac{1}{2}, -3$

$y = -\frac{2}{3}x^{-1} + Ax^{\frac{1}{2}} + Bx^{-3}$ 2

[d] Solve the initial value problem $2x^2y'' + 7xy' - 3y = \frac{4}{x}$, $\quad y(1) = \frac{4}{3}$, $\quad y'(1) = -\frac{1}{3}$.

$y(1) = -\frac{2}{3} + A + B = \frac{4}{3} \longrightarrow A + B = 2$

$y' = \frac{2}{3}x^{-2} + \frac{1}{2}Ax^{-\frac{1}{2}} - 3Bx^{-4}$

$y'(1) = \frac{2}{3} + \frac{1}{2}A - 3B = -\frac{1}{3} \longrightarrow \frac{1}{2}A - 3B = -1$

$A - 6B = -2$

$A + B = 2$ $\longrightarrow -7B = -4$

$B = \frac{4}{7}$

$A = \frac{10}{7}$

$y = -\frac{2}{3}x^{-1} + \frac{10}{7}x^{\frac{1}{2}} + \frac{4}{7}x^{-3}$

Find the general solutions of the following homogeneous linear differential equations. **SCORE: _____ / 5 PTS**

[a] $16y'' - 24y' + 9y = 0$

$16r^2 - 24r + 9 = 0$ $\frac{1}{2}$

$(4r-3)^2 = 0$

$r = \frac{3}{4}, \frac{3}{4}$ $\frac{1}{2}$

$y = Ae^{\frac{3}{4}x} + Bxe^{\frac{3}{4}x}$ 1

[b] $4x^2y'' + 12xy' + 5y = 0$

$4r^2 + 8r + 5 = 0$ 1

$r = \frac{-8 \pm \sqrt{64-80}}{8} = \frac{-8 \pm 4i}{8}$

$= -1 \pm \frac{1}{2}i$ 1

$y = Ax^{-1}\cos(\frac{1}{2}\ln x) + Bx^{-1}\sin(\frac{1}{2}\ln x)$ 1

Find the general solution of the homogeneous linear differential equation $2y''' - 3y'' + 18y' + 10y = 0$. **SCORE: _____ / 6 PTS**

$2r^3 - 3r^2 + 18r + 10 = 0$

$r = \pm \dfrac{1, 2, 5, 10}{1, 2} = \pm 1, 2, 5, 10, \frac{1}{2}, \frac{5}{2}$

2 SIGN CHANGES → 2 OR 0 POSITIVE ROOTS

$-2r^3 - 3r^2 - 18r + 10 = 0$

1 SIGN CHANGE → 1 NEGATIVE ROOT

$$\begin{array}{r|rrrr} -1 & 2 & -3 & 18 & 10 \\ & & -2 & 5 & -23 \\ \hline & 2 & -5 & 23 & -13 \end{array}$$

ALTERNATING, NO ROOTS < −1

$$\begin{array}{r|rrrr} \boxed{-\frac{1}{2}} & 2 & -3 & 18 & 10 \\ & & -1 & 2 & -10 \\ \hline & \boxed{2 \quad -4 \quad 20} & & & 0 \end{array}$$ ✓

$(r + \frac{1}{2})(2r^2 - 4r + 20) = 0$

$2(r + \frac{1}{2})(r^2 - 2r + 10) = 0$

$r = \dfrac{2 \pm \sqrt{4 - 40}}{2} = \dfrac{2 \pm 6i}{2} = \boxed{1 \pm 3i}$

① POINT EACH

$y = \boxed{C_1 e^{-\frac{1}{2}x}}$ $\boxed{+ C_2 e^{x} \cos 3x}$ $\boxed{+ C_3 e^{x} \sin 3x}$

$y_1 = e^{-x}$ is a solution of $xy'' + (2x+3)y' + (x+3)y = 0$. **SCORE: _____ / 8 PTS**

Find a second linearly independent solution.

$y_2 = ve^{-x}$

$y_2' = v'e^{-x} - ve^{-x}$

$y_2'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}$

1 POINT EACH UNLESS OTHERWISE NOTED

$xy_2'' + (2x+3)y_2' + (x+3)y_2$

$= xe^{-x}v'' - 2xe^{-x}v' + xe^{-x}v$ $\frac{1}{2}$

$+ (2x+3)e^{-x}v' - (2x+3)e^{-x}v$ $\frac{1}{2}$

$+ (x+3)e^{-x}v$ $\frac{1}{2}$

$= xe^{-x}v'' + 3e^{-x}v' = 0$

$xv'' + 3v' = 0$

$u = v'$ $\frac{1}{2}$ $xu' + 3u = 0$

$x\frac{du}{dx} + 3u = 0$

$\frac{1}{2}$ $\int \frac{1}{u}\,du = \int -\frac{3}{x}\,dx$

$\ln|u| = -3\ln|x|$

$u = x^{-3}$

$v' = x^{-3}$

$\frac{1}{2}$ $v = -\frac{1}{2}x^{-2}$

$y_2 = x^{-2}e^{-x}$